Q1 B be brownian motion

$f: \mathbb{R} \to \mathbb{R}$ $f(x) = (x-k)_+$

(a) $u(t,x) = E[e^{-r(T-t)} f(B_T) \mid B_t = x]$

$= E[e^{-r(T-t)} (B_T - k)_+ \mid B_t = x]$

$= e^{-r(T-t)} E[(B_T - k)_+ \mid B_t = x]$

$= e^{-r(T-t)} E[(B_T - B_t + x - k)_+ \mid B_t = x]$ $\quad B_T - B_t + x - k \geq 0 \Rightarrow B_T - B_t \geq k - x$

Since $B_T - B_t \sim N(0, T-t)$ $\quad$ Define $y = B_T - B_t$

$= \int_{k-x}^{\infty} e^{-r(T-t)} (y + x - k) \frac{1}{\sqrt{2\pi(T-t)}} e^{\frac{-y^2}{2(T-t)}} dy$.

$= \frac{1}{\sqrt{2\pi(T-t)}} e^{-r(T-t)} \underbrace{\int_{k-x}^{\infty} y e^{\frac{-y^2}{2(T-t)}} dy}_{①}$

$+ e^{-r(T-t)} \frac{1}{\sqrt{2\pi(T-t)}} (x-k) \underbrace{\int_{k-x}^{\infty} e^{\frac{-y^2}{2(T-t)}} dy}_{②}$

① $\int_{k-x}^{\infty} y e^{\frac{-y^2}{2(T-t)}} dy = \int_{k-x}^{\infty} e^{\frac{-y^2}{2(T-t)}} d\frac{y^2}{2}$

$= \int_{k-x}^{\infty} -(T-t)\, d\left(e^{\frac{-y^2}{2(T-t)}}\right)$

$= -(T-t)\left(0 - e^{\frac{-(k-x)^2}{2(T-t)}}\right)$

$= (T-t) e^{\frac{-(k-x)^2}{2(T-t)}}$

② $\frac{1}{\sqrt{2\pi(T-t)}} \int_{k-x}^{\infty} e^{\frac{-y^2}{2(T-t)}} dy$ $\quad$ Define $z = \frac{y}{\sqrt{T-t}}$

$= \frac{1}{\sqrt{2\pi}} \int_{\frac{k-x}{\sqrt{T-t}}}^{\infty} e^{-\frac{z^2}{2}} dz$

$= 1 - \Phi\left(\frac{k-x}{\sqrt{T-t}}\right)$

$u(t,x) = e^{-r(T-t)} \left[\sqrt{\frac{T-t}{2\pi}} e^{\frac{-(k-x)^2}{2(T-t)}} + (x-k)\left(1 - \Phi\left(\frac{k-x}{\sqrt{T-t}}\right)\right)\right]$

(b) $\partial_t u(t,x) = r e^{-r(T-t)} \left[\sqrt{\frac{T-t}{2\pi}} e^{\frac{-(k-x)^2}{2(T-t)}} + (x-k)\left(1 - \Phi\left(\frac{k-x}{\sqrt{T-t}}\right)\right)\right]$

$+ e^{-r(T-t)} \left[\frac{1}{2\sqrt{2\pi}} (-(T-t))^{-\frac{1}{2}} e^{\frac{-(k-x)^2}{2(T-t)}} + \sqrt{\frac{T-t}{2\pi}} e^{-\frac{(k-x)^2}{2(T-t)}} \left(\frac{(k-x)^2}{2} \frac{1}{(T-t)^2}\right)\right]$

$+ e^{-r(T-t)} \left[(x-k) P_z\left(\frac{k-x}{\sqrt{T-t}}\right) \frac{k-x}{2} (T-t)^{-\frac{3}{2}}\right]$

$= r u(t,x) + e^{-r(T-t)} \left[e^{\frac{-(k-x)^2}{2(T-t)}} \left(\frac{1}{2}(T-t)^{-\frac{3}{2}} \frac{1}{\sqrt{2\pi}} (k-x)^2 - \frac{1}{2\sqrt{2\pi(T-t)}} - \frac{1}{2}(T-t)^{-\frac{3}{2}} \frac{1}{\sqrt{2\pi}} (k-x)^2\right)\right.$

$= r u(t,x) + e^{-r(T-t)} \left[e^{\frac{-(k-x)^2}{2(T-t)}} \frac{1}{2\sqrt{2\pi(T-t)}}\right]$

$$\partial_x u(t,x) = \sqrt{\frac{T-t}{2\pi}}\, e^{-r(T-t)} \; e^{\frac{-(K-x)^2}{2(T-t)}} \frac{1}{(T-t)} (K-x)$$

$$+ e^{-r(T-t)} \left[\left(1 - \Phi\left(\frac{K-x}{\sqrt{T-t}}\right) + (x-K)\left(-P_Z\left(\frac{K-x}{\sqrt{T-t}}\right)\frac{-1}{\sqrt{T-t}}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi(T-t)}}\, e^{-r(T-t)} \; (K-x)\, e^{\frac{-(K-x)^2}{2(T-t)}}$$

$$+ e^{-r(T-t)} \left[1 - \Phi\left(\frac{K-x}{\sqrt{T-t}}\right) + \frac{x-K}{\sqrt{T-t}} P_Z\left(\frac{K-x}{\sqrt{T-t}}\right)\right]$$

$$= e^{-r(T-t)} \left(1 - \Phi\left(\frac{K-x}{\sqrt{T-t}}\right)\right) = e^{-r(T-t)} \; \Phi\left(\frac{x-K}{\sqrt{T-t}}\right)$$

$$\partial^2_{xx} u(t,x) = e^{-r(T-t)} \; P_Z\left(\frac{K-x}{\sqrt{T-t}}\right)\frac{-1}{\sqrt{T-t}}$$

$$= e^{-r(T-t)} \frac{1}{\sqrt{2\pi(T-t)}} \; e^{\frac{-(K-x)^2}{2(T-t)}}$$

(c) prove $\partial_t u(t,x) + \frac{1}{2}\partial^2_{xx} u(t,x) - r u(t,x)$

$$r u(t,x) + e^{-r(T-t)} \left(e^{\frac{-(K-x)^2}{2(T-t)}} \left(- \frac{1}{2\sqrt{2\pi(T-t)}} \right) \right.$$

$$+ \frac{1}{2} e^{-r(T-t)} \frac{1}{\sqrt{2\pi(T-t)}} \; e^{\frac{-(K-x)^2}{2(T-t)}} \quad - r u(t,x)$$

$$= e^{-r(T-t)} \left(\frac{-1}{2\sqrt{2\pi(T-t)}} \; e^{\frac{-(K-x)^2}{2(T-t)}} + \frac{1}{2\sqrt{2\pi(T-t)}} \; e^{\frac{-(K-x)^2}{2(T-t)}} \right)$$

$$= 0$$

Q2 (a) $S_t = S_0 \exp((r-\frac{1}{2}\sigma^2)t + \sigma B_t^Q)$

(b) $E^Q[e^{-rT} 1_{\{S_T \geq K\}}]$

$= e^{-rT} E^Q[1_{\{S_0 \exp(r-\frac{1}{2}\sigma^2)T + \sigma B_T^Q \geq K\}}]$

$= e^{-rT} E^Q[\frac{B_T^Q}{\sqrt{T}} \geq \frac{\ln(\frac{K}{S_0}) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}]$

✓ $= e^{-rT} \Phi\left(\frac{\ln(\frac{S_0}{K}) + (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$

(c) $E^Q[\underbrace{e^{-rT}(S_T 1_{\{S_T \leq K_1\}}}_{①} + \underbrace{\frac{K_1}{K_1-K_2}(S_T-K_2) 1_{\{K_1 < S_T \leq K_2\}}}_{②} + \underbrace{(S_T-K_2) 1_{\{S_T > K_2\}}}_{③}]$

① $E^Q[e^{-rT} S_T 1_{\{S_T \leq K_1\}}]$

$= e^{-rT} E^Q[S_T 1_{\{\frac{B_T^Q}{\sqrt{T}} \leq \frac{\ln(\frac{K_1}{S_0}) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\}}]$

$= e^{-rT} E^Q[S_0 \exp[(r-\frac{1}{2}\sigma^2)T + \sigma B_T^Q] 1_{\{\frac{B_T^Q}{\sqrt{T}} \leq \frac{\ln(\frac{K_1}{S_0}) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\}}]$

$= S_0 e^{-\frac{1}{2}\sigma^2 T} E^Q[e^{\sigma B_T^Q} 1_{\{\frac{B_T^Q}{\sqrt{T}} \leq \frac{\ln(\frac{K_1}{S_0}) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\}}]$

$= S_0 e^{-\frac{1}{2}\sigma^2 T} E^Q[e^{\frac{\sigma B_T^Q \sqrt{T}}{\sqrt{T}}} 1_{\{\frac{B_T^Q}{\sqrt{T}} \leq \frac{\ln(\frac{K_1}{S_0}) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\}}]$

$= S_0 e^{-\frac{1}{2}\sigma^2 T} \int_{-\infty}^{\frac{\ln(\frac{K_1}{S_0}) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} e^{\sigma\sqrt{T}x} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx$

$= S_0 \int_{-\infty}^{\frac{\ln(\frac{K_1}{S_0}) - (r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} \, dx$

$= S_0 \int_{-\infty}^{\frac{\ln(\frac{K_1}{S_0}) - (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$ def $z = x - \sigma\sqrt{T}$

$= S_0 \Phi\left(\frac{\ln(\frac{K_1}{S_0}) - (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$

② $E^Q\left[e^{-rT}\frac{K_1}{K_1-K_2}(S_T-K_2)\mathbb{1}_{\{K_1<S_T\le K_2\}}\right]$

$$= e^{-rT}\frac{K_1}{K_1-K_2}E^Q\left[\left(S_0\exp\left((r-\tfrac{1}{2}\sigma^2)T+\sigma B_T^Q\right)-K_2\right)\mathbb{1}_{\{K_1<S_T\le K_2\}}\right]$$

$$= S_0 e^{-\frac{1}{2}\sigma^2 T}\frac{K_1}{K_1-K_2}E^Q\left[e^{\sigma B_T^Q}\mathbb{1}_{\{K_1<S_T\le K_2\}}\right] - e^{-rT}\frac{K_1}{K_1-K_2}E^Q\left[K_2\mathbb{1}_{\{K_1<S_T\le K_2\}}\right]$$

$$= S_0\frac{K_1}{K_1-K_2}\left[\Phi\Big(\underbrace{\frac{\ln(\frac{K_2}{S_0})-(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{\lambda_1(K_2)}\Big)-\Phi\Big(\underbrace{\frac{\ln(\frac{K_1}{S_0})-(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{\lambda_1(K_1)}\Big)\right]$$

$$- e^{-rT}\frac{K_1K_2}{K_1-K_2}\left[\Phi\Big(\underbrace{\frac{\ln(\frac{K_2}{S_0})-(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{\lambda_2(K_2)}\Big)-\Phi\Big(\underbrace{\frac{\ln(\frac{K_1}{S_0})-(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{\lambda_2(K_1)}\Big)\right]$$

$$= S_0\frac{K_1}{K_1-K_2}\left(\Phi(\lambda_1(K_2))-\Phi(\lambda_1(K_1))\right) - e^{-rT}\frac{K_1K_2}{K_1-K_2}\left(\Phi(\lambda_2(K_2))-\Phi(\lambda_2(K_1))\right)$$

③ $E^Q\left[e^{-rT}(S_T-K_2)\mathbb{1}_{\{S_T>K_2\}}\right]$

$$= S_0 E^Q\left[S_T\mathbb{1}_{\{S_T>K_2\}}\right] - e^{-rT}K_2E^Q\left[\mathbb{1}_{\{S_T>K_2\}}\right]$$

$$= S_0\,\Phi\left(\frac{\ln(\frac{S_0}{K_2})+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - e^{-rT}K_2\,\Phi\left(\frac{\ln(\frac{S_0}{K_2})+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

Q3. $S_0 = 100 \quad \sigma = 0.3 \quad K = 95 \quad r = 0.02 \quad T = 0.75$

$$d_1 = \frac{\ln\left(\frac{100}{95}\right) + \left(0.02 + \frac{0.3^2}{2}\right) \times 0.75}{0.3 \times \sqrt{3/4}} \approx 0.385$$

$$d_2 = d_1 - 0.3 \times \sqrt{0.75} \approx 0.125$$

$N(d_1) \approx 0.6498$

$N(d_2) \approx 0.5497$

$$C = 100 \times 0.6498 - 95 \times e^{-0.015} \times 0.5497$$

$$\approx 13.536$$